# 1 Why I gag on Eq.(7.3.4)

Let's start with Eqs.(7.3.1–3) using a simpler notation in which the change in the action is

$$\delta I = i \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial(\Psi^\ell(x))} \mathcal{F}^\ell(x)\epsilon(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell(x))} \partial_\mu \left( \mathcal{F}^\ell(x)\epsilon(x) \right) \right]. \tag{1}$$

If the fields obey the dynamical equations, then $\delta I = 0$. The global symmetry is that

$$0 = i\epsilon \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial(\Psi^\ell(x))} \mathcal{F}^\ell(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell(x))} \partial_\mu \mathcal{F}^\ell(x) \right]. \tag{2}$$

The dynamical equations imply that

$$0 = i\epsilon \int d^4x \left\{ \left[ \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell(x))} \right) \right] \mathcal{F}^\ell(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell(x))} \partial_\mu \mathcal{F}^\ell(x) \right\} \tag{3}$$

or that

$$0 = i\epsilon \int d^4x \; \partial_\mu \left[ \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell(x))} \right) \mathcal{F}^\ell(x) \right]. \tag{4}$$

But I don't see any conserved current here. It's rather that the integral of the divergence of the current

$$J^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell(x))} \mathcal{F}^\ell(x) \tag{5}$$

vanishes

$$0 = \int d^4x \; \partial_\mu J^\mu(x). \tag{6}$$

Let's return to eq.(1):

$$\begin{aligned} \delta I \;\; = \;\; & i \int d^4x \left[ \left( \frac{\partial \mathcal{L}}{\partial(\Psi^\ell(x))} \mathcal{F}^\ell(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell(x))} \left( \partial_\mu \mathcal{F}^\ell(x) \right) \right) \epsilon(x) \right. \\ & \left. + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell(x))} \mathcal{F}^\ell(x) \partial_\mu \epsilon(x) \right]. \end{aligned} \tag{7}$$

The first line of this equation does not vanish because it is weighted by $\epsilon(x)$. So I don't see why Eq.(7.3.4) should be true.

## 2 Internal Symmetry

Stationarity of the action

$$I = \int d^4x \; \mathcal{L}(x), \tag{8}$$

implies the dynamical equations

$$\frac{\partial \mathcal{L}}{\partial(\Psi^\ell(x))} = \partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_a \Psi^\ell(x))} \right) \tag{9}$$

summed over $a = 0, 1, 2, 3$.

Let's use the notation

$$\Psi^\ell_{,a}(x) = \frac{\partial \Psi^\ell(x)}{\partial x^a}. \tag{10}$$

We'll assume that the Lagrange density is a function of the fields and their first derivatives

$$\mathcal{L}(x) = \mathcal{L}(\Psi^\ell(x), \Psi^\ell_{,a}(x), x). \tag{11}$$

Suppose it is unchanged

$$\Big(\mathcal{L}(\Psi^\ell(x), \Psi^\ell_{,a}(x), x)\Big)' = \mathcal{L}(\Psi'^\ell(x), \Psi'^\ell_{,a}(x), x) = \mathcal{L}((\Psi^\ell(x), \Psi^\ell_{,a}(x)), x) \tag{12}$$

when we scramble the fields

$$\Psi'^\ell(x) = \Psi^\ell(x) + d\Psi^\ell(x) \tag{13}$$

at any point of space-time. Such an invariance is called an **internal symmetry**.

Often the tiny change $d\Psi^\ell(x)$ is something like

$$d\Psi^\ell(x) = M^\ell_m \Psi^m(x) \tag{14}$$

in which $M$ is a matrix, and (as usual) we sum over the repeated index $m$. This is a **global internal symmetry**. But if the Lagrange density is invariant under something like

$$d\Psi^\ell(x) = M^\ell_m(x) \Psi^m(x) + N^\ell(x) \tag{15}$$

then the symmetry is said to be **local**. Nature prefers **local internal symmetries**: all the interactions of the Standard Muddle are required to make

various symmetries local. Theories with local symmetry are called **gauge theories**.

It follows from the internal symmetry (12) that

$$\begin{aligned} 0 = \delta\mathcal{L}(x) &= \mathcal{L}(\Psi'^{\ell}(x), \Psi'^{\ell}_{,a}(x)) - \mathcal{L}((\Psi^{\ell}(x), \Psi^{\ell}_{,a}(x))) \\ &= \frac{\partial\mathcal{L}}{\partial(\Psi^{\ell}(x))} d\Psi^{\ell}(x) + \frac{\partial\mathcal{L}}{\partial(\Psi^{\ell}_{,a}(x))} (d\Psi^{\ell}(x))_{,a}. \end{aligned} \tag{16}$$

Lagrange's equations (9) now imply that

$$0 = \left( \frac{\partial\mathcal{L}}{\partial(\Psi^{\ell}_{,a}(x))} \right)_{,a} d\Psi^{\ell}(x) + \frac{\partial\mathcal{L}}{\partial(\Psi^{\ell}_{,a}(x))} (d\Psi^{\ell}(x))_{,a}. \tag{17}$$

Thus the current

$$j^a(x) = \frac{\partial\mathcal{L}}{\partial(\Psi^{\ell}_{,a}(x))} d\Psi^{\ell}(x) \tag{18}$$

is conserved

$$0 = j^a_{,a}(x). \tag{19}$$

So internal symmetries imply conserved currents. If Lagrange's density has an internal symmetry, then the theory has a conserved current $j^a(x)$. The conserved current $j^a(x)$ implies that a quantity

$$Q = \int d^3x\, j^0(x) \tag{20}$$

is conserved:

$$\begin{aligned} \partial_0 Q &= \int d^3x\, j^0_{,0}(x) = -\int d^3x\, j^i_i(x) \\ &= \text{surface integral at spatial infinity } = 0. \end{aligned} \tag{21}$$

QED gives us an important example. With $F_{ab} = \partial_a A_b - \partial_b A_a$ the Maxwell field tensor and $\Psi$ the Dirac field of the electron, its Lagrange density is

$$\mathcal{L}(x) = -\frac{1}{4} F_{ab}(x) F^{ab}(x) - \bar{\Psi}(x)(\gamma^a[\partial_a + ieA_a(x)] + m)\Psi(x) \tag{22}$$

and it is invariant under the global transformation

$$\Psi'(x) = e^{ie\lambda}\Psi(x) \qquad \text{or} \qquad d\Psi(x) = ie\lambda\Psi(x) \tag{23}$$

for constant $\lambda$ ( and $e$). The formula (18) gives the conserved current apart from the constant factor $\lambda$

$$j^a(x) = \frac{\partial \mathcal{L}}{\partial(\Psi^\ell_{,a}(x))} \frac{d\Psi^\ell(x)}{\lambda} = -ie\bar{\Psi}(x)\gamma^a\Psi(x) \tag{24}$$

and the conserved quantity is the electric charge

$$Q = \int d^3x\, j^0(x) = -e \int d^3x\, \bar{\Psi}(x) i\gamma^0 \Psi(x) = -e \int d^3x\, \Psi^\dagger(x)\Psi(x). \tag{25}$$

The minus sign arises because $-e$ is the charge of the electron and has been with us since politicians last understood science, that is, since the time of Benjamin Franklin. Here we used the relation that defines the gamma-matrices

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} = 2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

and the definition $\bar{\Psi} = \Psi^\dagger\beta = \Psi^\dagger i\gamma^0$. These give $\bar{\Psi}\gamma^0 i = \Psi^\dagger i\gamma^0 i\gamma^0 = \Psi^\dagger$.

Homework problems: Show that for the action density (22) of QED, Lagrange's equations are Maxwell's equations

$$j^a + \partial_b F^{ba} = 0 \tag{27}$$

and Dirac's equation

$$(\gamma^a[\partial_a + ieA_a(x)] + m)\,\Psi(x) = 0. \tag{28}$$

Also, show that the action density (22) is invariant under the global gauge transformation (23).

Suppose the Lagrange density is invariant under a local symmetry like (15). Then for every pair of functions $M^\ell_m(x)$ and $N^\ell(x)$, there's a conserved current

$$J^a_{MN}(x) = \frac{\partial \mathcal{L}}{\partial(\Psi^\ell_{,a}(x))} \left[M^\ell_m(x)\Psi^\ell(x) + N^\ell(x)\right] \tag{29}$$

and if these functions $M^\ell_m(x)$ and $N^\ell(x)$ are non-zero only in their own spatial spheres, then these currents and their conserved quantities

$$Q_{MN} = \int d^3x\ J^0_{MN}(x) \tag{30}$$

would seem to be all different. So a gauge theory apparently has an infinite number of conserved quantities. (Wrong!)

To see what's wrong with this argument, we return to QED whose action density $\mathcal{L}(x)$ is invariant under the **gauge transformation**

$$\begin{aligned} \Psi'(x) &= \exp(ie\lambda(x)\Psi(x)) \\ A'_b(x) &= A_b(x) - \partial_b\lambda(x) \end{aligned} \tag{31}$$

in which $\lambda(x)$ now depends upon the space-time point $x$. In this case, the current (29) is

$$J^a_\lambda = -ie\lambda\bar{\Psi}\gamma^a\Psi - F^{ba}\partial_b\lambda = \lambda j^a - F^{ba}\partial_b\lambda \tag{32}$$

and so the conserved quantity for each localized distribution $\lambda(x)$ is

$$Q_\lambda = \int d^3x\ J^0_\lambda(x) = \int d^3x\ (\lambda j^0 - F^{i0}\partial_i\lambda). \tag{33}$$

But after integrating the second term by parts, dropping the surface term, we find that these all vanish

$$Q_\lambda = \int d^3x\ \lambda(j^0 + \partial_i F^{i0}) = 0 \tag{34}$$

because of Gauss's law

$$j^0 + \partial_i F^{i0} = 0. \tag{35}$$

which is the $a = 0$ Maxwell equation (27).

Homework problem: Show that the action density (22) is invariant under the local gauge transformation (31).

# 3 Translational Invariance

Usually, we assume that the Lagrange density depends upon the space-time point $x$ only thru the dependence of fields $\Psi_\ell(x)$ upon $x$. Thus

$$\mathcal{L}_{,a}(x) = \frac{\partial\mathcal{L}}{\partial(\Psi^\ell(x))}\Psi^\ell_{,a}(x) + \frac{\partial\mathcal{L}}{\partial(\Psi^\ell_{,b}(x))}\Psi^\ell_{,ba}(x). \tag{36}$$

If we now invoke Lagrange's equations (9), then we find that

$$\mathcal{L}_{,a}(x) = \left(\frac{\partial\mathcal{L}}{\partial(\Psi^\ell_{,b}(x))}\right)_{,b}\Psi^\ell_{,a}(x) + \frac{\partial\mathcal{L}}{\partial(\Psi^\ell_{,b}(x))}\Psi^\ell_{,ba}(x). \tag{37}$$

Thus we have

$$0 = \left[ \delta^b_a \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\Psi^\ell_{,b}(x))} \Psi^\ell_{,a}(x) \right]_{,b} . \tag{38}$$

Thus there are four currents

$$T^b{}_a = \delta^b_a \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\Psi^\ell_{,b}(x))} \Psi^\ell_{,a}(x) \tag{39}$$

that are conserved

$$0 = T^b{}_{a,b} \tag{40}$$

because of translational invariance. With both indices lowered, this energy-momentum tensor is

$$T_{ba} = \eta_{ba} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\Psi^{\ell,b}(x))} \Psi^\ell_{,a}(x). \tag{41}$$

The four conserved quantities are the four-momentum operators

$$P_a = \int d^3x \, T^0{}_a = \int d^3x \left( \delta^0_a \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\Psi^\ell_{,0}(x))} \Psi^\ell_{,a}(x) \right). \tag{42}$$

The hamiltonian is

$$H = P^0 = -P_0 = \int d^3x \left( \frac{\partial \mathcal{L}}{\partial(\Psi^\ell_{,0}(x))} \Psi^\ell_{,0}(x) - \mathcal{L} \right). \tag{43}$$

The momentum is

$$P^i = P_i = -\int d^3x \, \frac{\partial \mathcal{L}}{\partial(\Psi^\ell_{,0}(x))} \Psi^\ell_{,i}(x). \tag{44}$$

# 4 Lorentz Invariance

In general, the energy-momentum tensor $T^{ab}$ is not symmetric, but when it is—as it is for a theory of scalar fields—we may use it to make six new conserved currents from the tensor

$$\mathcal{M}^{abc} = x^b T^{ac} - x^c T^{ab} \tag{45}$$

which by construction is anti-symmetric in its second and third indices. For fixed $b$ & $c$, the divergence of this tensor vanishes

$$\mathcal{M}^{abc}_{,a} = (x^b \, T^{ac} - x^c \, T^{ab})_{,a} = \delta^b_a \, T^{ac} + x^b \, T^{ac}_{,a} - \delta^c_a \, T^{ab} - x^c \, T^{ab}_{,a}. \tag{46}$$

because by (40), the divergences $T^{ac}_{,a}$ and $T^{ab}_{,a}$ vanish, leaving

$$\mathcal{M}^{abc}_{,a} = T^{bc} - T^{cb} = 0 \tag{47}$$

which is zero when the tensor $T^{ab}$ is symmetric. So the vanishing of a symmetric energy-momentum tensor implies the conservation of six currents $\mathcal{M}^{abc}$. The six conserved quantities

$$J^{bc} = \int d^3x\, \mathcal{M}^{0bc} \tag{48}$$

are the six generators of the Lorentz group.

When second and third indices are spatial, then the conserved quantities are the generators of rotations

$$J^{ij} = \int d^3x\, \mathcal{M}^{0ij} \tag{49}$$

that is, they are the angular-momentum operators. Before working out what the other three "boost" operators are, let's find a symmetric energy-momentum tensor $\Theta^{ab}$. The energy-momentum tensor that is the source of the gravitational field must be symmetric.

Under a Lorentz transformation

$$x'^a = \Lambda^a{}_b\, x^b \tag{50}$$

the four-volume element is unchanged

$$d^4x' = d^4x. \tag{51}$$

If $U(\Lambda)$ is the unitary operator that implements a Lorentz transformation $\Lambda$, then a field $\Psi^\ell(x)$ transforms as

$$\Psi'^\ell(x) = U(\Lambda)\Psi^\ell(x)U^{-1}(\Lambda) = D^\ell_m(\Lambda^{-1})\Psi^m(\Lambda x) \tag{52}$$

apart from a possible gauge transformation in the case of massless spin-one (or spin-two) fields.

If the Lorentz transformation is infinitesimal

$$\begin{aligned} \Lambda^a{}_b &= \delta^a{}_b + \omega^a{}_b \\ \omega_{ab} &= -\omega_{ba} \end{aligned} \tag{53}$$

then the change in the field $\Psi^\ell$ is

$$\delta\Psi^\ell = \frac{i}{2}\omega^{ab}\left(\mathcal{J}_{ab}\right)^\ell_m\ \Psi^m \tag{54}$$

apart from the change in its argument from $x$ to $x' = \Lambda x$. The (script-J) matrices $\mathcal{J}_{ab}$ generate the Lie algebra of the representation of the Lorentz group

$$[\mathcal{J}_{mn}, \mathcal{J}_{rs}] = i\mathcal{J}_{rn}\eta_{ms} - i\mathcal{J}_{sn}\eta_{mr} - i\mathcal{J}_{ms}\eta_{nr} + i\mathcal{J}_{mr}\eta_{ns} \tag{55}$$

appropriate to the field $\Psi^\ell$.

The matrices $\mathcal{J}_{mn}$ are all zero for a field of spin-zero. For a Dirac field, they are commutators of gamma matrices

$$\mathcal{J}^{mn} = -\frac{i}{4}[\gamma^m, \gamma^n] \tag{56}$$

as noted in Eq.(5.4.6). For a covariant vector field, they are

$$\left(\mathcal{J}_{rs}\right)_k^{\ \ell} = -i\eta_{rk}\,\delta^\ell_s + i\eta_{sk}\,\delta^\ell_r \tag{57}$$

because, apart from a possible gauge transformation, the change in a covariant vector field is

$$\delta V_a = \omega_a^{\ b}\,V_b. \tag{58}$$

The derivative of a field transforms like (54) but with an extra covariant vector index

$$\delta\Psi^\ell_{,k} = \frac{i}{2}\omega^{ab}\left(\mathcal{J}_{ab}\right)^\ell_{\ m}\ \Psi^m_{,k} + \omega_k^{\ c}\Psi^\ell_{,c}. \tag{59}$$

Under a Lorentz transformation, the action density $\mathcal{L}(x)$ transforms as a scalar

$$\mathcal{L}'(x) = U(\Lambda)\mathcal{L}(x)U^{-1}(\Lambda) = \mathcal{L}(\Lambda x) \tag{60}$$

and so the changes proportional to $\omega$ in (54) & (59) must cancel

$$0 = \frac{\partial\mathcal{L}}{\partial\Psi^\ell}\,\frac{i}{2}\omega^{ab}\left(\mathcal{J}_{ab}\right)^\ell_m\ \Psi^m + \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\left[\frac{i}{2}\omega^{ab}\left(\mathcal{J}_{ab}\right)^\ell_{\ m}\ \Psi^m_{,k} + \omega_k^{\ c}\Psi^\ell_{,c}\right]. \tag{61}$$

Since $\omega_k^{\ c} = \eta_{ka}\omega^{ab}$, the anti-symmetry of $\omega^{ab}$ implies that

$$\begin{aligned}\omega_k^{\ c}\Psi^\ell_{,c} &= \eta_{ka}\omega^{ab}\Psi^\ell_{,b} = \frac{1}{2}\eta_{ka}\omega^{ab}\Psi^\ell_{,b} - \frac{1}{2}\eta_{ka}\omega^{ba}\Psi^\ell_{,b} \\ &= \frac{1}{2}\eta_{ka}\omega^{ab}\Psi^\ell_{,b} - \frac{1}{2}\eta_{kb}\omega^{ab}\Psi^\ell_{,a} = \frac{1}{2}\omega^{ab}\left(\eta_{ka}\Psi^\ell_{,b} - \eta_{kb}\Psi^\ell_{,a}\right).\end{aligned} \tag{62}$$

Setting the coefficient of $\omega^{ab}$ equal to zero gives

$$0 = i\frac{\partial\mathcal{L}}{\partial\Psi^\ell}\,\left(\mathcal{J}_{ab}\right)^\ell_m\,\Psi^m + i\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\mathcal{J}_{ab}\right)^\ell{}_m\,\Psi^m_{,k} + \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\eta_{ka}\Psi^\ell_{,b} - \eta_{kb}\Psi^\ell_{,a}\right). \quad (63)$$

The dynamical equations (9) now imply

$$0 = i\left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\mathcal{J}_{ab}\right)^\ell{}_m\,\Psi^m\right]_{,k} + \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\eta_{ka}\Psi^\ell_{,b} - \eta_{kb}\Psi^\ell_{,a}\right) \quad (64)$$

or

$$0 = i\left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\mathcal{J}_{ab}\right)^\ell{}_m\,\Psi^m\right]_{,k} + \frac{\partial\mathcal{L}}{\partial\Psi^{\ell,a}}\,\Psi^\ell_{,b} - \frac{\partial\mathcal{L}}{\partial\Psi^{\ell,b}}\,\Psi^\ell_{,a}. \quad (65)$$

Using our formula (41) for the naive energy-momentum tensor, we may write this as

$$0 = i\left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\mathcal{J}_{ab}\right)^\ell{}_m\,\Psi^m\right]_{,k} - T_{ab} + T_{ba}. \quad (66)$$

Thus

$$T_{ba} = T_{ab} - i\left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\mathcal{J}_{ab}\right)^\ell{}_m\,\Psi^m\right]_{,k} \quad (67)$$

and so we may symmetrize $T_{ab}$ by forming the symmetric combination

$$T_{\{ab\}} = \frac{1}{2}\left(T_{ab} + T_{ba}\right) = T_{ab} - \frac{i}{2}\left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\mathcal{J}_{ab}\right)^\ell{}_m\,\Psi^m\right]_{,k}. \quad (68)$$

But we must keep the divergence $T^{\{ab\}}_{,a}$ equal to $T^{ab}_{,a}$ which vanishes. We can do that if we can make the quantity inside the square brackets anti-symmetric in $a$ and $k$. Belinfante solved this puzzle by subtracting two terms that together are symmetric in $ab$ (so as not to spoil the symmetry in $ab$). The first of these terms makes the first two of the three terms inside the square brackets anti-symmetric in $a$ and $k$; the second of these terms is anti-symmetric in $a$ and $k$ because $\mathcal{J}^{ka}$ is:

$$\Theta^{ab} = T^{ab} - \frac{i}{2}\left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,k}}\,\left(\mathcal{J}^{ab}\right)^\ell{}_m\,\Psi^m - \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,a}}\,\left(\mathcal{J}^{kb}\right)^\ell{}_m\,\Psi^m - \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,b}}\,\left(\mathcal{J}^{ka}\right)^\ell{}_m\,\Psi^m\right]_{,k}. \quad (69)$$

Belinfante's energy-momentum tensor is symmetric

$$\Theta^{ab} = \Theta^{ba} \quad (70)$$

and has the same vanishing divergence as does the naive energy-momentum tensor

$$\Theta^{ab}_{,a} = T^{ab}_{,a} = 0. \tag{71}$$

Thus we can use $\Theta^{ab}$ as the source of gravitation and also to make the six conserved currents

$$\mathcal{M}^{abc} = x^b \Theta^{ac} - x^c \Theta^{ab} \tag{72}$$

as we did in the case (45) in which $T^{ab}$ is itself symmetric. As before (47), these six currents are conserved

$$\mathcal{M}^{abc}_{,a} = 0 \tag{73}$$

because $\Theta^{ab}$ is symmetric and has no divergence $\Theta^{ab}_{,a} = 0$.

Of the six conserved quantities, the three that are most useful are the generators $J^{ij}$ of rotations (49). These angular-momentum operators seem very complex

$$\begin{aligned} J^{ij} &= \int d^3x\, \mathcal{M}^{0ij} = \int d^3x \left( x^i \Theta^{0j} - x^j \Theta^{0i} \right) \\ &= \int d^3x \left\{ x^i T^{0j} - x^j T^{0i} \right. \\ &- \frac{ix^i}{2} \left[ \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,k}} \left( \mathcal{J}^{0j} \right)^\ell{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,0}} \left( \mathcal{J}^{kj} \right)^\ell{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,j}} \left( \mathcal{J}^{k0} \right)^\ell{}_m \Psi^m \right]_{,k} \\ &+ \left. \frac{ix^j}{2} \left[ \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,k}} \left( \mathcal{J}^{0i} \right)^\ell{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,0}} \left( \mathcal{J}^{ki} \right)^\ell{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,i}} \left( \mathcal{J}^{k0} \right)^\ell{}_m \Psi^m \right]_{,k} \right\}. \end{aligned} \tag{74}$$

But the integration is spatial. And the quantities inside the square brackets vanish for $k = 0$ because the first two terms cancel and because $\mathcal{J}^{00} = 0$ due to its anti-symmetry

$$0 = \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,0}} \left( \mathcal{J}^{0j} \right)^\ell{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,0}} \left( \mathcal{J}^{0j} \right)^\ell{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,j}} \left( \mathcal{J}^{00} \right)^\ell{}_m \Psi^m \tag{75}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,0}} \left( \mathcal{J}^{0i} \right)^\ell{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,0}} \left( \mathcal{J}^{0i} \right)^\ell{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \Psi^\ell_{,i}} \left( \mathcal{J}^{00} \right)^\ell{}_m \Psi^m. \tag{76}$$

So if we integrate by parts and drop the surface terms, then we find

$$J^{ij} = \int d^3x \left\{ x^i T^{0j} - x^j T^{0i} \right. \tag{77}$$

$$
\begin{aligned}
&+ \quad \frac{i}{2}\left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,i}}\left(\mathcal{J}^{0j}\right)^\ell{}_m\Psi^m - \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,0}}\left(\mathcal{J}^{ij}\right)^\ell{}_m\Psi^m - \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,j}}\left(\mathcal{J}^{i0}\right)^\ell{}_m\Psi^m\right] \\
&- \quad \frac{i}{2}\left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,j}}\left(\mathcal{J}^{0i}\right)^\ell{}_m\Psi^m - \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,0}}\left(\mathcal{J}^{ji}\right)^\ell{}_m\Psi^m - \frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,i}}\left(\mathcal{J}^{j0}\right)^\ell{}_m\Psi^m\right]\Bigg\}.
\end{aligned}
$$

The like-colored terms cancel each other, and so

$$
J^{ij} = \int d^3x\,\left[x^iT^{0j} - x^jT^{0i} - i\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,0}}\left(\mathcal{J}^{ij}\right)^\ell{}_m\Psi^m\right]. \tag{78}
$$

By using our formula (41) for the naive energy-momentum tensor, we may express the angular-momentum operators as

$$
J^{ij} = \int d^3x\,\left[-x^i\frac{\partial\mathcal{L}}{\partial(\Psi^\ell_{,0})}\Psi^{\ell,j} + x^j\frac{\partial\mathcal{L}}{\partial(\Psi^\ell_{,0})}\Psi^{\ell,i} - i\frac{\partial\mathcal{L}}{\partial\Psi^\ell_{,0}}\left(\mathcal{J}^{ij}\right)^\ell{}_m\Psi^m\right] \tag{79}
$$

or more simply as

$$
J^{ij} = \int d^3x\,\frac{\partial\mathcal{L}}{\partial(\Psi^\ell_{,0})}\left[-x^i\Psi^{\ell,j} + x^j\Psi^{\ell,i} - i\left(\mathcal{J}^{ij}\right)^\ell{}_m\Psi^m\right]. \tag{80}
$$

**Canonical fields** are ones whose time derivatives appear in the action density. Weinberg often writes such fields as $Q^n$. The canonical momentum $P_n$ of a canonical field $Q^n$ is

$$
P_n(x) = \frac{\partial\mathcal{L}(x)}{\partial Q^n_{,0}(x)} \tag{81}
$$

and it does not vanish identically—as it would were there no $Q^\ell_{,0}$ in $\mathcal{L}$. Canonical fields are easier to deal with than non-canonical fields because they do not present constraints like $P^\ell = 0$. It it therefore fortunate that the angular-momentum operators (80) involve only canonical fields

$$
J^{ij} = \int d^3x\,P_n\left[-x^iQ^{n,j} + x^jQ^{n,i} - i\left(\mathcal{J}^{ij}\right)^n{}_mQ^m\right]. \tag{82}
$$